Embedding Graphs on the Torus with Monodromy

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Motivation

One question in topological graph theory is whether or not we can embed graphs on certain surfaces. For tori, this question becomes whether or not we have a valid Belyĭ pair (E, β), where E is an elliptic curve and β is a Belyĭ map. Once we have a valid Belyĭ pair, we want to compute its monodromy group to understand the dessin d'enfant corresponding to its Belyĭ map.
We wish to:

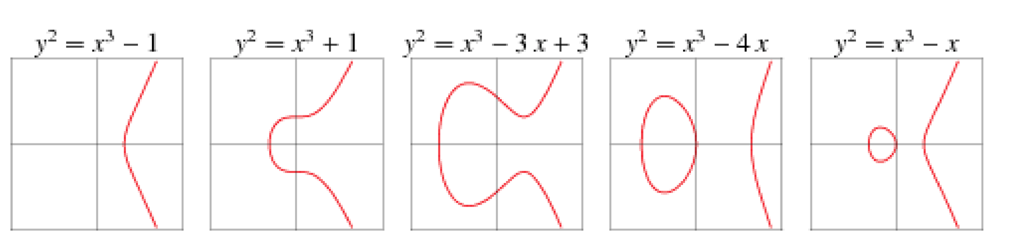
- Understand monodromy groups of Dessins d'Enfants on the torus
- Compile a database of toroidal Dessins d'Enfants and Belyi pairs

Background

- **Elliptic Curves** An elliptic curve E is a set

$$E(\mathbb{C}) = \left\{ (x:y:z) \in \mathbb{P}^2(\mathbb{C}) \;\middle|\; \begin{array}{c} y^2z + a_1x\,y\,z + a_3y\,z^2 \\ = x^3 + a_2x^2z \\ + a_4xz^2 + a_6z^3 \end{array} \right\}$$

for complex numbers a_1, a_3, a_2, a_4, a_6.



Examples of elliptic curves

- The surface defined by an Elliptic curve over the complex numbers is equivalent to a torus.
- **Belyĭ Map**
  A Belyĭ Map is a rational function $\beta : E(\mathbb{C}) \to \mathbb{P}^1(\mathbb{C})$ with at most 3 critical values, which we assume to be $\{0, 1, \infty\}$. Here $\mathbb{P}^1(\mathbb{C})$ is the Complex Projective Line.

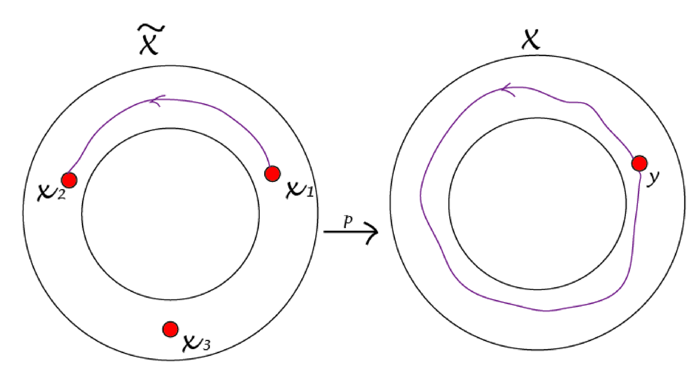
  Some examples include:

$$\beta(x,y) = \frac{y+1}{2} \quad \text{for} \quad E : y^2 = x^3 + 1$$
$$\beta(x,y) = \frac{(y - x^2 - 17x)^3}{2^{14}y} \quad \text{for} \quad E : y^2 + 15xy + 128y = x^3$$
$$\beta(x,y) = \frac{(x-5)y + 16}{32} \quad \text{for} \quad E : y^2 = x^3 + 5x + 10$$

- **Dessins d'Enfant** A bipartite graph is a graph whose vertices will be composed of 2 disjoint sets, in this case represented by 2 different colors: black and white. Given a Belyĭ map β, its corresponding Dessin d'Enfant is a bipartite graph of black and white vertices given by:
  - $\beta^{-1}(0)$ = Black Vertices
  - $\beta^{-1}(1)$ = White Vertices
  - $\beta^{-1}([0,1])$ = Edges.

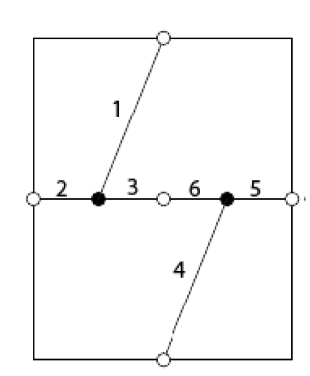
Monodromy

- **Covering Spaces**
  - Let X be a topological space. A covering space of X consists of a topological space $\tilde{X}$ and a map $p : \tilde{X} \to X$ such that for each $x \in X$, there exists an open neighborhood U of x such that $p^{-1}(U)$ is the disjoint union of open sets, each of which is mapped homeomorphically onto U by p. The degree of the map is defined as $|p^{-1}(x)|$.
    - A Belyĭ map acts as a covering map on $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.
- **Monodromy Groups**
  - Let $f : X \to Y$ be a covering map of degree d. Fixing a point $y \in Y$, we can define an action of $\pi_1(Y, y)$ on the set $f^{-1}(y)$ as follows:
  - Let $x_1, x_2, \ldots x_d$ be points above y and $\gamma \in \pi_1(Y, y)$ be a loop. By the unique lifting property of covering space, there is a unique path γ_i starts at each x_i that lifts γ. Let $x_{\sigma(i)}$ be the end point of γ_i. It must be a point above y. Then $i \to \sigma(i)$ is a permutation of the $x_i's$. This gives an action of $\pi_1(Y, y)$ on the points of the preimage of y.
  - This action is called monodromy action and is equivalent to a group homomorphism $\beta : \pi_1(Y, y) \to S_d$. The image of β is called monodromy group.
- **Example** $\tilde{X}$ is a covering space of the annulus X with covering map p such that $p(x_1) = p(x_2) = p(x_3) = y$.



The monodromy group of this covering space of the anulus is $Z_3 \subset S_3$.

- **Monodromy Groups of Dessins d'Enfants**



We can compute the generators of the monodromy group of a dessin on the torus as follows:

- Label the edges of the graph with the numbers $1, \ldots, |E|$.
- Let σ_0 be the product of cycles given by listing the edges we meet when tracing a small counterclockwise loop around each black vertex.
- Let σ_1 be the product of cycles given by listing the edges we meet when tracing a small counterclockwise loop around each white vertex.
- Choose σ_∞ such that $\sigma_0\sigma_1\sigma_\infty = 1$.
- Let $G = \langle \sigma_0, \sigma_1 \rangle \subset S_{|E|}$.
- In this case, $\sigma_0 = (123)(456)$, $\sigma_1 = (14)(25)(36)$, $\sigma_\infty = (162435)$, and $G \cong \mathbb{Z}_6 \subset S_6$.

Toroidal Degree Sequences

- Let D be a dessin. Its degree sequence $\mathcal{D}$ is defined to be the set $\{B, W, F\}$, where B, W and F are sets of numbers, defined as follows:
  - $B = \{e_b$ |b is a black vertex, and e_b is the number of edges adjacent to it$\}$
  - $W = \{e_w$ |w is a white vertex, and e_w is the number of edges adjacent to it$\}$
  - $F = \{e_f$ |f is a face, and e_f is the number of white vertices adjacent to it$\}$
  - The degree sequence of a toroidal dessin of degree d must satisfy $|B| + |W| + |F| = d$.
  - We call a degree sequence regular if each black vertex, each white vertex, and each face has the same degree. Thus the degree sequence of a toroidal graph is of the form

$$D = \{\underbrace{\{k, \ldots, k\}}_{d/k}, \underbrace{\{l, \ldots, l\}}_{d/l}, \underbrace{\{m, \ldots, m\}}_{d/k}\}.$$

  - Thus k, l, m and d must satisfy $\frac{d}{k} + \frac{d}{l} + \frac{d}{m} = d$. Cancelling out d, we get the relationship $\frac{1}{k} + \frac{1}{l} + \frac{1}{m} = 1$.
  - Up to permutation, the only solutions are $(k, l, m) = (3,2,6), (4,2,4), (3,3,3)$.

Infinite Families

There is an infinite family of Dessins d'Enfants corresponding to each choice of $(k, l, m) = (3,2,6), (4,2,4), (3,3,3)$.

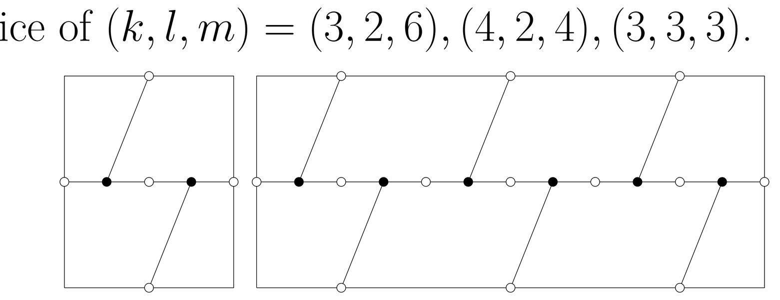


Figure 1: The $n = 1$ and $n = 3$ cases of

$$D_{(2,3,6)n} = \{\underbrace{\{3, \ldots, 3\}}_{2n}, \underbrace{\{2, \ldots, 2\}}_{3n}, \underbrace{\{6, \ldots, 6\}}_{n}\}$$

The monodromy group of $D_{(2,3,6)n}$ is $\mathbb{Z}_6 \rtimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

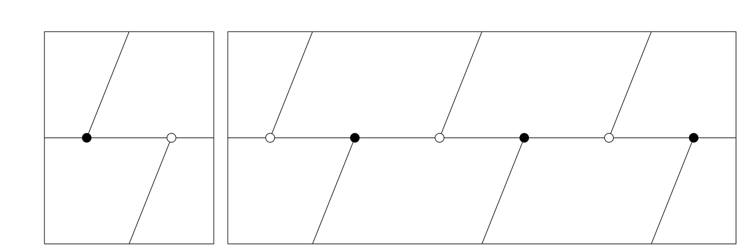


Figure 2: The $n = 1$ and $n = 3$ cases of

$$D_{(3,3,3)n} = \{\{3, \ldots, 3\}, \{3, \ldots, 3\}, \{3, \ldots, 3\}\}$$

The monodromy group of $D_{(3,3,3)n}$ is $\mathbb{Z}_3 \rtimes (\mathbb{Z}_n \times \mathbb{Z}_n)$.

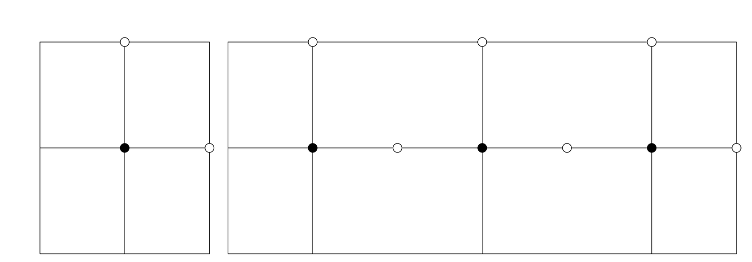


Figure 3: The $n = 1$ and $n = 3$ cases of

$$D_{(4,2,4)n} = \{\{4, \ldots, 4\}, \{2, \ldots, 2\}, \{4, \ldots, 4\}\}$$

. The monodromy group of $D_{(2,3,6)n}$ is $\mathbb{Z}_4 \rtimes (\mathbb{Z}_n \times \mathbb{Z}_n)$

Belyĭ maps for these dessins can be given by the composition of an $n-$isogeny and the Belyĭ map for the $n = 1$ case.

Database of Belyĭ Pairs

For each $N \in \mathbb{N}$, our database consists of the following:

- The degree sequence D, a partition of N.
- A Belyĭ pair (E, β) where β has degree N (if applicable)
- The monodromy group associated with D (if applicable)

Algorithm for Computing Elements in the Database

For a given $N \in \mathbb{N}$, we must do the following:

1. Find all partitions $P = \{e_1, \ldots, e_m\}$ of N such that $e_i \in \mathbb{Z}^+$ and $e_1 + \ldots + e_m = N$.
2. Choose three partitions P_0, P_1, P_∞. Keep the triple only if $N = |P_0| + |P_1| + |P_\infty|$.
3. For a degree sequence $D = \{P_0, P_1, P_\infty\}$, computing its Belyĭ pair requires us to solve a system of equations to find the coefficients of the elliptic curve and Belyĭ map.
4. Once we have said Belyĭ pair, we can compute its dessin d'enfant in the manner described above.
5. Computing the monodromy group from that dessin d'enfant also occurs in the manner described above.

Future Work

Although for the infinite family of Dessins d'Enfant described above we can construct Belyĭ maps by recursively computing the composition of an n-isogeny and the first Belyĭ map, it is unknown how the the monodromy groups of these maps behave under composition of functions.

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Acknowledgments

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